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Phase Transitions on Markovian Bipartite Graphs—an Application of the Zero-range Process

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We analyze the existence and the size of the giant component in the stationary state of a Markovian model for bipartite multigraphs, in which the movement of the edge ends on one set of vertices of the bipartite graph is a zero-range process, the degrees being static on the other set. The analysis is based on approximations by independent variables and on the results of Molloy and Reed for graphs with prescribed degree sequences. The possible types of phase diagrams are identified by studying the behavior below the zero-range condensation point. As a specific example, we consider the so-called Evans interaction. In particular, we examine the values of a critical exponent, describing the growth of the giant component as the value of the dilution parameter controlling the connectivity is increased above the critical threshold. Rigorous analysis spans a large portion of the parameter space of the model exactly at the point of zero-range condensation. These results, supplemented with conjectures supported by Monte Carlo simulations, suggest that the phenomenological Landau theory for percolation on graphs is not broken by the fluctuations.

KEY WORDS: Random graph; giant component; zero-range process; condensation.

1. INTRODUCTION

The common way of implementing time dependence in the models of random graphs is to couple the time and the number of edges in such a way that, starting with an empty graph with a given number of vertices, a new edge is added uniformly at random to one of the vacant edge locations at each time step. This procedure is known simply as the *graph process*.[(1)] During the past few years, a large variety of models generalizing

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the graph process have appeared in the physics literature (see ref. 2 for a review), most of them motivated by the structure of the Internet and the World Wide Web. One of the central ideas of these models is known, by the work of Barabási and Albert,[3] as preferential attachment. In the Barabási–Albert model new vertices are introduced one at a time and joined to one of the existing vertices chosen with probability proportional to the *degree* of the vertex, i.e. the number of edges already joined to it. This preferential attachment leads to what Price[4] calls cumulative advantage, and it can be used to construct so-called scale-free networks, i.e. graphs with power-law distributed degrees.

In the graph process and the Barabási–Albert model, the edges, once attached to the vertices, retain their positions *ad infinitum*. In the present article, we consider what happens if internal restructuring by rewiring of the graph with the existing edges dominates the dynamics, that is, the time scales associated with addition and removal of edges and vertices, have become very long as compared to the times between the rewiring events. The dynamics is taken to be the simplest possible capable of providing the cumulative advantage—namely of zero-range[5] so that the rate of a rewiring event, i.e. a jump of an edge end from one vertex to another, depends only on the degree of either the initial or the final vertex of the move. We focus on the connectivity properties of the graphs in the stationary state. The question about the existence and size of the giant component, a connected cluster occupying a macroscopic fraction of vertices, turns out particularly interesting because of the condensation phenomena induced by the zero-range processes. Closely related models have been studied on a phenomenological level by Palla *et al.*[6] and rigorous results for models of polymerization sharing the same kind of ideas have been obtained by Pittel *et al.*[7,8] A statistical mechanics approach to a rather general class of reversible graph-valued processes is presented in ref. 9.

We take bipartite multigraphs as the state space of the problem so that the system consists of external agents with static degrees and another set of vertices, for example collaboration events, on which the movement of the edge ends, coming from the set of agents, takes place. A process with unipartite states is obtained as a special case to our model, but its behavior is not as rich as in the bipartite version.

The structure of the article is as follows: In the following section, we give a precise definition of the model and discuss the reasons for choosing the particular state space. Section 3 reviews some fundamental results about zero-range processes and a Poisson approximation for the numbers of vertices of given degree is constructed. This is applied to subcritical zero-range processes in Section 4. In Section 4.1, two simple examples are given and general implications of the results for subcritical zero-range processes are discussed in Section 4.2. In Section 5, we concentrate on restructuring processes with the so-called Evans interaction. A detailed view of the phase diagram and the critical exponents is given in Subsection 5.1. The connection with a model of diluted scale-free networks is also established. Conclusions are made and some open problems presented in Section 6.

2. STATEMENT OF THE PROBLEM

We shall consider bipartite multigraphs with vertex sets W and V such that $|W| = L$ and $|V| = M_1 + M_2$ (see Fig. 1(a)). Here M_1 denotes the number of vertices of degree 1 and M_2 the number of vertices of degree 2 and these degrees are assigned to randomly chosen vertices in V independently of each other. This set up on V will be static during the time evolution. The total number of edges is $N = M_1 + 2M_2$ and we shall consider only cases with the densities

$$r := 2M_2/N \in [0, 1], \qquad \rho := N/L \in (0, \infty) \tag{1}$$

fixed on the passage to the limit $L \to \infty$.

Let us then discuss the dynamics. Initially, the N edge ends are chosen to be distributed uniformly and independently of each other on the vertices in W. The restructuring of the graph takes place in continuous time by jumps of the edge ends on this set. We take the process on W to be a zero-range process[5,10] with M_1 particles of type 1 (edges to vertices of degree 1 in V) and $2M_2$ particles of type 2 (edges to vertices of degree 2 in V), so that a vertex $w \in W$ loses a randomly chosen edge end after an exponentially distributed time, where the parameter of the exponential distribution is a function of the degree of w. More precisely, if $w \in W$ is an endvertex of X_w^1 edges to vertices of degree 1 and of X_w^2 edges to vertices of degree 2, the exponential rate is given by $g(X_w^1 + X_w^2)$, where g is a given positive function with bounded increments, and the edge end to jump is one of those leading to vertices of degree 1 with probability $X_w^1/(X_w^1 + X_w^2)$ and one of those leading to vertices of degree 2 otherwise. Immediately after leaving the vertex w, the edge end joins one of the other vertices according to a symmetric, irreducible transition matrix $(P_{w,z})$, $w, z \in W$. The stationary distribution of this zero-range process can be found explicitly and will be studied in the next section. Clearly, the stationary distribution of the corresponding bipartite multigraph-valued process is such that all the states with the same degree sequence are equiprobable.

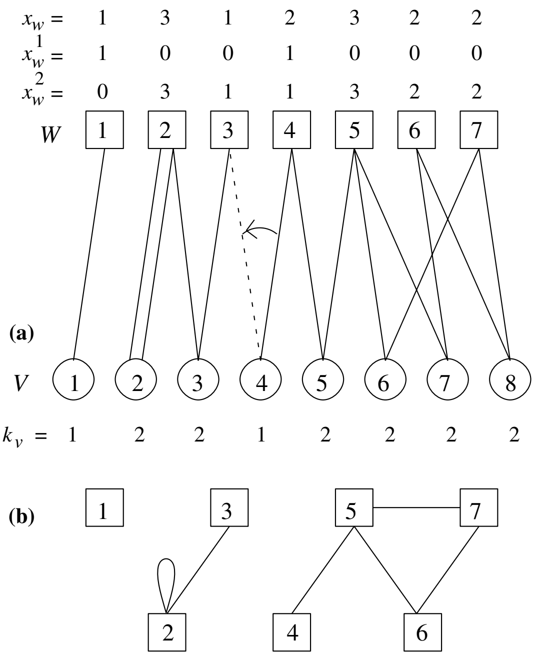

Fig. 1. (a) Example of a bipartite multigraph with vertex sets V and W. The static degrees of $v \in V$ are denoted by K_v and the evolving degrees of $w \in W$ by $X_w = X_w^1 + X_w^2$ with X_w^1 and X_w^2 the numbers of edges leading to vertices of degree one and two, respectively. The arrow and the dotted line show a possible transition. (b) The projection of the bipartite graph in (a) to the vertex set W.

In this article, we study the structure of the graphs in the stationary state of the process. In particular, we concentrate on the properties of the projection of the bipartite graph, also called one-mode network,[11] on the set W, obtained by adding an edge between two vertices in W if one can be reached from the other by traversing two edges on the original bipartite graph, see Fig. 1(b). In other words, the projection tells us how the vertices in W are mutually connected by the vertices of the other type. The particular questions we ask are then, when does a giant component, a connected subgraph of size proportional to L, exist on the projection and precisely of how many vertices such a component is made?

There are several reasons for making the particular choices concerning the state space. First of all, the zero-range dynamics is easily implemented in this setting and it contains the unipartite model as the special case $M_1 = 0$. The bipartite structure really brings something more to the theory: The M_1 vertices of degree 1 are dangling ends from the point of view of connectivity, and do not appear on the projection graph, but they do have influence on the jump rates of the zero-range process. The parameter $r=2M_2/N$ controls the amount of this "dark matter" and lowering its value while keeping $\rho=N/L$ fixed corresponds to diluting the set of edges on the projection. Therefore it will be referred to as a dilution parameter. Varying the values of r and ρ simultaneously in such a way that their product remains constant can be used to sweep the phase diagram of the underlying particle system, characterized by the density ρ alone, without changing the density of edges on the projection. In addition, the restriction of the static degrees to at most 2 allows us to use the usual configuration model,[1] and therefore the results of Molloy and Reed,[12,13] directly without going through the proofs in the bipartite setting. Moreover, we believe that the extension that allows higher but bounded degrees or even degree distributions with exponential tails does not change the qualitative features of the phase diagrams or the values of the critical exponents.

3. ZERO-RANGE PROCESSES AND POISSON APPROXIMATIONS

Let the dynamics of the edge ends on the set W be as described in the previous section and let Ω_{L,M_1,M_2} be the set of ordered partitions of two numbers M_1 and $2M_2$ both in L non-negative parts. Then, for $(\eta,\xi)\in\Omega_{L,M_1,M_2}$, one shows by verifying that the detailed balance condition holds that the stationary (canonical) distribution of the zero-range process is given by

$$\mu_{L,M_1,M_2}(\eta,\xi)=\frac{1}{Z_\mu(L,M_1,M_2)}\prod_{w=1}^{L}\binom{\eta_w+\xi_w}{\eta_w}\frac{1}{g!(\eta_w+\xi_w)}, \tag{2}$$

where $g!(n)=\prod_{k=1}^{n}g(k)$ is a generalized factorial with the convention that $g!(0)=1$ and $Z_\mu(L,M_1,M_2)$ is a normalization factor (canonical partition function). Let $D_{m,n}$ denote the number of vertices in W with exactly m edges leading to vertices of degree 1 and n edges leading to vertices of degree 2. Then (2) implies an image measure

$$\sigma_{L,M_1,M_2}(d)=\frac{1}{Z_\sigma(L,M_1,M_2)}\prod_{m=0}^{M_1}\prod_{n=0}^{M_2}\frac{1}{d_{m,n}!}\left(\binom{m+n}{m}\frac{1}{g!(m+n)}\right)^{d_{m,n}} \tag{3}$$

for the joint random variable $D=(D_{m,n})$ in the stationary regime. In Eq. (3), d is such that the three conditions $\sum_{m=0}^{M_1}\sum_{n=0}^{2M_2}d_{m,n}=L$,

$\sum_{m=0}^{M_1}\sum_{n=0}^{2M_2}(m+n)d_{m,n}=M_1+2M_2$ and $\sum_{m=0}^{M_1}\sum_{n=0}^{2M_2}md_{m,n}=M_1$ are satisfied. Otherwise σ_{L,M_1,M_2} vanishes.

Our purpose is to get a grip of the random variables $D_n{:=}\sum_{m=0}^{M_1}D_{m,n}$, i.e. the numbers of vertices of degree n on the projection to W. More formally, $\mathcal{D}=(D_n)$ is the asymptotic degree sequence of Molloy and Reed[12,13] in our problem and we have to show that, with high probability,[14] we get an asymptotic degree sequence $\mathcal{D}$, which is in their sense well-behaved (see Appendix A for definitions and discussion). Because of the constraints on d in Eq. (3), the random variables $D_{m,n}$ are not independent, and it would be difficult to extract the behavior of D_n from the measure σ_{L,M_1,M_2} directly. Next we are going to construct a sequence of independent variables that is shown to be a good approximation for the original sequence. In other words, we shall study the problem in the grand canonical ensemble and prove that the canonical and grand canonical ensembles are equivalent.

For zero-range processes, approximations by independent variables have been considered by Großkinsky *et al.*[15] and Jeon *et al.*[16] In their highly influential article, Jeon *et al.* were able to give a rigorous proof for the behavior of the largest cluster not only for subcritical processes and for the cases with $b>3$ in what is sometimes called Evans interaction,[10] $g(k)=1+b/k+O(k^{-(1+\delta)})$, but also for a large class of vanishing rates. Within this class, which essentially consists of functions that vanish faster than $\exp(-c\sqrt{\log k})$, a single vertex would hold all but of the order $Lg(L)$ edge ends in our model, so that the projection of the graph would be in a flower-like state with one vertex having many self-loops. We choose to study the vanishing cases no further. By restricting ourselves to the cases with the interaction function g bounded away from zero, we also avoid the truncation of the series involved in the grand canonical analysis. The phase diagram of the Evans model was also discussed by Großkinsky *et al.* and heuristic arguments, supported by simulation results, for the dynamics of condensation were given. The results of these articles concerning the Evans interaction will be applied in later sections. The general method and the validity of approximations by independent variables is discussed in a delightful paper of Arratia and Tavaré.[17]

Let now g be bounded away from zero and let $C_{m,n}$, $m,n\in\mathbb{Z}_+$, be independent Poisson$(\lambda_{m,n}(\alpha,\gamma,\phi))$-distributed random variables, where

$$\lambda_{m,n}(\alpha,\gamma,\phi)=\frac{1}{Z(\phi)}\binom{m+n}{m}\frac{\alpha(1-\gamma)^m\gamma^n\phi^{m+n}}{g!(m+n)},\tag{4}$$

$$Z(\phi)=\sum_{k=0}^{\infty}\frac{\phi^k}{g!(k)},\tag{5}$$

and α, γ and ϕ are real parameters whose values will be determined later. Here Z is the grand canonical partition function. Its radius of convergence, which we shall denote by Φ, is positive because g is bounded away from zero. In order to avoid cumbersome expressions, we set

$$\begin{aligned} A_1 &= \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n} = L \right\}, \\ A_2 &= \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+n) C_{m,n} = N \right\}, \\ A_3 &= \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m C_{m,n} = M_1 \right\}. \end{aligned} \tag{6}$$

Then one can show by a direct calculation that $(D_{0,0}, \ldots, D_{M_1,2M_2}) = (C_{0,0}, \ldots | A_1 \cap A_2 \cap A_3)$ in law and, because of cancellation in the conditional probabilities, independent of the values of α, γ and ϕ. So $(C_{m,n})$ really is a candidate for a good approximation. The construction suggests the following for the approximative system:

1. A direct calculation shows that the system size $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n}$ is a Poisson(α)-distributed random variable.

2. Again by a direct computation, for $t \in \mathbb{R}$,

$$E \exp\left(it \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+n) C_{m,n} \right) = E \left(\frac{Z(\phi e^{it})}{Z(\phi)} \right)^{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n}},$$

in that

$$\left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+n) C_{m,n} \middle| A_1 \right) = \sum_{w=1}^{L} Y_w$$

in law, with the variables Y_w independent and distributed according to the grand canonical distribution

$$\nu_\phi(k) := \frac{1}{Z(\phi)} \frac{\phi^k}{g!(k)}, \qquad k \in \mathbb{Z}_+. \tag{7}$$

In other words, ν_ϕ controls the vertex degrees locally. The expectation of a ν_ϕ-distributed variable will be denoted by $R(\phi)$:

$$R(\phi) := EY_w = \phi Z'(\phi)/Z(\phi). \tag{8}$$

The parameter ϕ is known as the fugacity.

3. In a similar manner one shows that

$$E\left[\exp\left(it\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} mC_{m,n}\right)\middle| A_1\right] = E\left(\gamma + (1-\gamma)e^{it}\right)^{\sum_{w=1}^{L} Y_w},$$

so that, given A_1 and A_2, the number of edges leading to vertices of degree equal to unity has the binomial distribution with parameters N and γ.

Now suppose that one would like to get an exponentially decaying upper bound for the probability of an event B concerning the dependent variables $(D_{m,n})$, and let $\mathcal{B}$ be the same event for the independent variables $(C_{m,n})$. Suppose further that the parameters α, γ and ϕ can be chosen in such a way that the probability of the conditioning event $A_1 \cap A_2 \cap A_3$ is not exponentially small. Then it suffices to get the upper bound for the independent case: $P(B) = P(\mathcal{B}|A_1 \cap A_2 \cap A_3) \leqslant P(\mathcal{B})/P(A_1 \cap A_2 \cap A_3)$. The use of this conditioning device is nicely illustrated in the articles of Corteel *et al.*[18] and Jeon *et al.*[16]

How to best choose the values of the parameters? The point 1 in the list above tells that the approximative system size $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{m,n}$ is sharply peaked at α with a standard deviation of $\sqrt{\alpha}$ so that the simplest choice is to match the parameter with the size of the canonical system L. Notice that the choice is not unique because terms of the order $o(L)$ could be added without affecting the behavior in the large system size limit. The points 2 and 3 in the list show that, with the appropriate conditionings, the same type of argument applies to the two other random variables in the conditions (6) as well. We thus set

$$\alpha = L, \qquad \gamma = 2M_2/N = r \tag{9}$$

and, if possible,

$$R(\phi) = N/L = \rho. \tag{10}$$

Clearly, the first two equations can always be fulfilled but in the third one the case

$$\rho_c := \lim_{\phi \to \Phi} R(\phi) < \rho \tag{11}$$

marks a condensation transition. Roughly speaking, the best one can do in that case is to set $\phi = \Phi$, so that locally one observes degrees distributed according to ν_Φ, just like in a system with density ρ_c. Then the rest of the mass, that is $(\rho - \rho_c)L$ edges, must hide in the $o(L)$-sets of vertices since those edge ends are not picked up by the approximating measure. It was proven in refs. 15 and 16 that this really is the true picture and, furthermore, the authors of ref. 15 claim that the set of $o(L)$ vertices with the rest of the mass is in fact a single vertex for a large class of rate functions. Especially, the Evans model, which we shall study in Section 5, belongs to this class. As a by-product of the discussion above, we get something really important to the theory. From the point of view of our application the condensates for vanishing rates and the ones bounded away from zero are completely different: In the first case, the components on the flower-like projection graph are very small, while in the second, the condensate on a single vertex implies the existence of a giant component.

Recall that $D_n := \sum_{m=0}^{M_1} D_{m,n}$ is the number of vertices of degree n on the projection, so that $C_n := \sum_{m=0}^{\infty} C_{m,n}$ is an independent approximation for this quantity. We define

$$\nu_\phi^r(n) = \frac{1}{Z(\phi)} \sum_{k=n}^{\infty} \binom{k}{n} r^n (1-r)^{k-n} \frac{\phi^k}{g!(k)}, \tag{12}$$

which equals the expected value of C_n divided by the system size. To show that the conditions of Molloy and Reed are satisfied, we first prove that the fluctuations C_n around $\nu_\phi^r(n)L$ are not too large. The following inequality is rather easily obtained for any $\epsilon > 0$ using Chernoff's bounds (see e.g. ref. 19):

$$P\left(\left|\frac{C_n}{L} - \nu_\phi^r(n)\right| > \epsilon\right) \leqslant \exp\left(-\epsilon L f(\epsilon/\nu_\phi^r(n))\right), \tag{13}$$

where $f(x) = ((1+x)\log(1+x) - x)/x$, with $x > 0$, is strictly increasing. Using here the trivial bound $f(\epsilon/\nu_\phi^r(n)) \geqslant f(\epsilon)$ and the fact that $\epsilon f(\epsilon) \geqslant$

$(1+\epsilon)\epsilon/(1+\epsilon/2)-\epsilon=\epsilon^2/(2+\epsilon)$ yields that, with high probability for any $\delta>0$,

$$C_n=\nu_\phi^r(n)L+o(L^{1/2+\delta}). \tag{14}$$

In order to get the estimates for the original sequence, one has to bound $P(A_1\cap A_2\cap A_3)$ from below.

4. SUBCRITICAL ZERO-RANGE PROCESSES

Let us now derive a subexponential lower bound for the probability of the conditioning event in the subcritical cases $\rho_c:=\lim_{\phi\to\Phi}\phi Z'(\phi)/Z(\phi)>\rho$. Notice first that with the choices made in Eq. (9) and using the results for the variables in A_1 and A_3, the probability of A_1 is bounded below by $c_1/\sqrt{L}$ for some constant c_1 and also $P(A_3|A_1\cap A_2)\geqslant c_3/\sqrt{L}$. So we have to bound the probability of A_2 given A_1. By subcriticality and from the fact that $\phi Z'(\phi)/Z(\phi)$ is strictly increasing it follows that there is a unique $\phi<\Phi$ for every value of the density such that Eq. (10) is satisfied and the distribution ν_ϕ therefore has an exponential tail. Therefore the usual local limit theorem[20] applies and the probability is at least $c_2/\sqrt{L}$. Thus $P(A_1\cap A_2\cap A_3)\geqslant cL^{-3/2}$ and the bound (14) for the fluctuations of C_n applies to D_n as well. Moreover, the tail of ν_ϕ^r is exponentially thin because $\nu_\phi^r(n)=O(\nu_\phi(n)/r)$ and ν_ϕ shows exponential decay. Also, since $\sum_{k\geqslant n}\nu_\phi(k)=O(c^n)$, for some $c\in(0,1)$ and large n, the largest degree is $O(\log L)$ with high probability. Now it is easy to check that, with high probability, the degree sequence (D_n) is such that the conditions of Molloy and Reed, reproduced in Appendix A, are satisfied.

Next we give the results for the subcritical cases. Let $\Phi>0$ and $\rho_c>\rho$ and define

$$r_c(\phi):=\frac{Z'(\phi)}{\phi Z''(\phi)}. \tag{15}$$

Then, the parts of Theorem 1 of ref. 12 concerning the existence of the giant component and Theorem 1 of ref. 13 state that, with high probability:

1. If $r<r_c(\phi)$, the largest component on the projection to W has at most of the order $\log^3 L$ vertices.

2. If $r>r_c(\phi)$, there is exactly one component on the projection with more than $T\log L$ vertices for some constant T, and the size of that com-

ponent is $L\Delta + o(L)$, where

$$\Delta = 1 - \frac{Z((1-\beta r)\phi)}{Z(\phi)} \tag{16}$$

and β is the positive solution to

$$\beta = 1 - \frac{Z'((1-\beta r)\phi)}{Z'(\phi)}. \tag{17}$$

Notice that, since $Z'((1-\beta r)\phi)$ is a convex function of β on $(0, 1)$, Eq. (17) has a unique positive solution exactly when $r > r_c(\phi)$.

We can also deduce the leading order of Δ as a function of $r - r_c(\phi)$: The expansion of the right-hand side of Eq. (17) for small β (all the derivatives of $Z(\phi)$ exist because $\phi < \Phi$) yields

$$\beta \approx \frac{2Z'(\phi)}{r^2\phi^2 Z'''(\phi)} \left(\frac{r}{r_c(\phi)} - 1\right) \tag{18}$$

as $r \to r_c$ from above, so that now expanding the right-hand side of Eq. (16) to the first order in β we have

$$\Delta \approx r\rho\beta \approx \frac{2\phi Z''(\phi)^2}{Z(\phi) Z'''(\phi)}(r - r_c(\phi)). \tag{19}$$

Thus the growth starts linearly. Furthermore, the uniqueness of the solution to Eqs. (16) and (17) allows us to calculate the leading order of Δ when the critical curve is crossed by increasing the density ρ with constant r. At the point $(\phi + \epsilon, r_c(\phi))$ with $\phi + \epsilon < \Phi$, we get

$$\Delta \approx -\frac{2\phi Z''(\phi)^2}{Z(\phi) Z'''(\phi)} r_c'(\phi)\epsilon \tag{20}$$

for the size of the giant component. Note that, in case of a finite radius of convergence Φ of the partition function, the $\phi \to \Phi$ limits of the amplitudes in expressions (19) and (20) depend crucially on the existence of the third moment of a ν_Φ-distributed random variable.

4.1. Two Examples

As a first example, let us study the case of non-interacting random walks. Then $g(k) = k$ and $Z(\phi) = \exp(\phi)$, so that from Eqs. (8), (10) and from the definition of the critical density of the zero-range process, Eq. (11), we get that $\phi = \rho$ and $\rho_c = \infty$. Also, from Eq. (15), it follows that $r_c(\rho) = 1/\rho$, for $\rho \geqslant 1$, and there is no phase transition for $\rho < 1$ (remember that $r = 2M_2/N \in [0, 1]$). Of course, by $r_c(\rho)$ we mean $r_c(\phi(\rho))$ with $\phi(\rho)$ the solution to Eq. (10). Setting $c := r\rho$ one recovers the results familiar from the model $G_{n,M=cn}$ of random graphs:[1] The phase transition occurs at $c = 1$ and the size of the giant component is $L\Delta + o(L)$ with Δ given by the solution to $\Delta = 1 - \exp(-c\Delta)$. Furthermore, Eq. (19) is consistent with the known expansion $\Delta = 2\epsilon - 8\epsilon^2/3 + O(\epsilon^3)$ for $c = 1 + \epsilon$.

In our second example we take the jump rates to be degree independent, $g(k) = 1$. Now the partition function has a finite radius of convergence, $\Phi = 1$, but still there is no condensation transition, so that $\rho_c = \infty$. In this case we have $r_c(\rho) = 1/(2\rho)$, for $\rho \geqslant 1/2$, and no transition for smaller densities. Quite surprisingly, the size of the giant component has a simple expression

$$\Delta = \frac{3}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{r\rho}},$$

from which, again in harmony with Eq. (19), we get $\Delta = 4\rho\epsilon/3 - 56(\rho\epsilon)^2/27 + O(\epsilon^3)$, for $r = 1/(2\rho) + \epsilon$.

4.2. General Remarks on the Phase Diagrams

The critical curve can be alternatively written in terms of the function $R(\phi)$ defined in Eq. (8):

$$r_c(\phi) = \left(R(\phi) - 1 + \phi \frac{R'(\phi)}{R(\phi)} \right)^{-1}, \tag{21}$$

from which it follows that

$$r_c(\rho) \leqslant \frac{1}{\rho - 1}. \tag{22}$$

The fact that the equation for the critical curve involves the second derivative of the partition function, or equivalently the first derivative of R, already tells that the phase diagram of the graph-valued process can be more complicated than for the zero-range process itself, in which the existence of the phase transition is simply dictated by the asymptotics of the ratio $Z'(\phi)/Z(\phi)$. Three different kinds of behavior can be extracted from the results (15) and (21):

1. When $R(\phi) \to \infty$ as $\phi \to \Phi$, i.e ρ_c is infinite, the critical curve decreases to zero as a function ρ in such a way that $r_c(\rho) > 0$ for ρ arbitrarily large. The high density asymptotic behavior of the curve is either $r_c(\rho) \sim 1/\rho$ or given by the nontrivial part in the denominator of (21). This point will be explored further shortly.

2. When the first derivative of the partition function converges as $\phi \to \Phi$, in that ρ_c is finite, but the second derivative diverges, we have a condensation transition, and the critical curve approaches zero as ρ tends to ρ_c from below. Therefore, a giant component exists at and above the zero-range critical point for any non-zero r. Near the condensation transition, where ϕ is just below Φ, the critical curve goes like

$$r_c(\rho) \approx \frac{\rho}{\phi(\rho) R'(\phi(\rho))}. \tag{23}$$

3. If also the second derivative of the partition function converges, $\lim_{\rho\to\rho_c} r_c(\rho)$ is non-zero. In fact, the values of the dilution parameter r being restricted to the unit interval, this case is further divided into two categories according to whether this non-zero limiting value is greater or less than 1. For $\lim_{\rho\to\rho_c} r_c(\rho) > 1$, no phase transition can be achieved by varying the dilution parameter r. In both cases, however, if the condensation occurs on a single vertex, we know that above ρ_c there is a giant component, but the theory developed so far tells nothing about its existence exactly at the zero-range critical point.

These three points illustrate the shape of the critical curve at the high density limit. We remark that, for small values of ρ, the curve is expected to pick up the features of the non-interacting case $g(k) = k$, provided that $\lim_{\rho\to\rho_c} r_c(\rho) < 1$. An example of a concrete phase diagram, with all the features discussed here, will be given in Section 5.

We saw that the interactions in the two examples of Section 4.1 produced critical curves inversely proportional to the density. Let us now deduce, for which functions g this proportionality holds, at least for densities high enough. For this purpose, set

$$r_c(\phi) = \frac{1}{aR(\phi)}, \tag{24}$$

with $a>0$, in (21). This yields a linearizable first order differential equation for R, with the initial condition $R'(0)=1/g(1)$, the solution to which is

$$R(\phi)=\frac{\phi}{g(1)+(1-a)\phi}, \tag{25}$$

where $\phi\in[0,g(1)/(a-1))$ for $a>1$ and $\phi\in[0,\infty)$ otherwise. Furthermore, solving for $Z(\phi)$ from Eq. (8), we get

$$Z(\phi)=\left(1+\frac{1-a}{g(1)}\phi\right)^{\frac{1}{1-a}}, \tag{26}$$

which has the expansion

$$Z(\phi)=1+\frac{\phi}{g(1)}+\sum_{m\geqslant 2}\frac{1}{m!}\left(\frac{\phi}{g(1)}\right)^m\prod_{k=2}^{m}[(k-1)a-(k-2)], \tag{27}$$

so that

$$g(k)=\begin{cases} g(1), & \text{for } k=1\\ \frac{g(1)k}{(k-1)a-(k-2)}, & \text{for } k\geqslant 2.\end{cases} \tag{28}$$

This form of interaction function covers a range of models, which we now classify for different values of a:

For $a=1$ or 2 we get the examples of Section 4.1.

For $1<a\neq 2$, the tail of $g(k)$, which determines the high density behavior, can be expanded in powers of $1/k$:

$$g(k)=\sum_{n=1}^{n_*}\delta_{n,k}\frac{(a-1)n}{(n-1)a-(n-2)}+\sum_{n=n_*+1}^{\infty}\delta_{n,k}\sum_{p=0}^{\infty}\left(\frac{a-2}{a-1}\right)^p\frac{1}{n^p}, \tag{29}$$

where $n_*=\max\{0,[\frac{a-2}{a-1}]\}$, and we have chosen $g(1)=a-1$ to make the radius of convergence equal to unity. To leading order in the large k limit, $g(k)$ is then given by

$$g(k)\sim\begin{cases} 1-b(a)/k, & \text{for } a\in(1,2),\\ 1+b(a)/k, & \text{for } a>2,\end{cases} \tag{30}$$

where $b(a)=|(a-2)/(a-1)|$.

In the first case of $1 < a < 2$, which interpolates between the cases of non-interacting edges and the degree independent rates, the range of b as a function of a is $\mathbb{R}_+$, so that the behavior

$$r_c(\rho) = \frac{1+b}{(2+b)\rho} \tag{31}$$

is expected to hold for all $b \in (0, \infty)$ in the high density limit for the interaction $g(k) = 1 - b/k + O(k^{-(1+\delta)})$.

The interesting case of $a > 2$ with decreasing interaction function is in ref. 15 referred to as the Evans interaction due to his work presented in ref. 10. Now the image of $\{a > 2\}$ under b is the interval $(0, 1)$—just a tiny part of the nontrivial parameter space $0 < b < \infty$. Thus, we predict

$$r_c(\rho) = \frac{1-b}{(2-b)\rho} \tag{32}$$

for high densities in the Evans model with $0 < b < 1$ and something else for the rest of the phase diagram. This model is known to exhibit a condensation transition for $b > 2$, in which case our critical curve $r_c(\rho)$ will hit zero at ρ_c. The region $1 < b \leqslant 2$ is therefore special—the critical curve is strictly above zero, but falls of faster than the first inverse power in ρ. This seems to be consistent with the observation of Großkinsky *et al.*[15] that a kind of precursor of condensation transition is present on this interval of the parameter space. The Evans model will be the topic of the next section.

In the region $0 < a < 1$, the cases with $a \in ((0, 1) \setminus \{1 - 1/(n+1) : n \in \mathbb{N}\})$ can be ruled out immediately, since they lead to interaction functions breaking the positivity assumption. So we are here left with the set $a \in \{1 - 1/(n+1) : n \in \mathbb{N}\}$. But this time the assumption on bounded increments is violated: $g(k)$ would be finite for $k \in \{1, \ldots, K\}$, where $K = 1/(1-a)$, and infinite for larger values of k. Thus we don't expect to find $1/(a\rho)$-behavior with $a < 1$ in our model. However, it is possible to construct processes that have stationary distributions corresponding to the partition functions (26) by altering the dynamics only slightly. This leads to the notion of generalized exclusion processes[21] and, in these processes, the jumps to sites already occupied by K particles are simply suppressed. Especially, the version with $K = 1$ is the simple exclusion process.[22–24] We would like to remark that, since for generalized exclusions the condition $\rho < K$ must be satisfied, the bound (22) would not be broken.

5. EVANS INTERACTION

In this section we take $g(k)=1+b/k$, $b\geqslant 0$, so that the partition function is given by a hypergeometric function,[25]

$$Z(\phi)=\sum_{k=0}^{\infty}\frac{(k!)^2\,\Gamma(1+b)}{\Gamma(k+1+b)}\cdot\frac{\phi^k}{k!}=F(1,1;1+b;\phi), \tag{33}$$

with the radius of convergence $\Phi=1$. Also, from the definition (8),

$$R(\phi)=\frac{\phi}{1+b}\cdot\frac{F(2,2;2+b;\phi)}{F(1,1;1+b;\phi)}, \tag{34}$$

from which the value of the critical density can be calculated[15]:

$$\rho_c=\begin{cases}\infty, & \text{for } b\leqslant 2\\ 1/(b-2), & \text{for } b>2.\end{cases} \tag{35}$$

The formula for the critical curve is, by the definition (15),

$$r_c(\phi)=\frac{2+b}{4\phi}\cdot\frac{F(2,2;2+b;\phi)}{F(3,3;3+b;\phi)}. \tag{36}$$

Let the size of the largest cluster in a zero-range process on L sites be denoted by Z_L^*. The following facts for Evans interaction are taken from ref. 16:

1. When $\rho<\rho_c$, that is $\phi<1$, the distribution ν_ϕ has an exponential tail, and Z_L^* is at most of the order $\log L$ with high probability.

2. At the critical point, that is $\phi=1$ and $b>2$, the distribution ν_ϕ has a heavy tail, $\nu_1(k)\sim k^{-b}$, and Z_L^* is exactly of the order $L^{1/(b-1)}$ with high probability. Furthermore,

$$P(A_2\,|\,A_1)\geqslant cL^{1-b}. \tag{37}$$

3. For $\rho>\rho_c$ and $b>3$ (this is needed in the proof),

$$\frac{Z_L^*}{L}\longrightarrow\rho-\rho_c \tag{38}$$

in probability as $L\to\infty$, and the size of the second largest cluster is $o(\sqrt{L}\log^2 L)$ with high probability.

The results of the previous sections cover the subcritical cases $\rho < \rho_c$ only. As already discussed, the case number 3 in the previous paragraph implies the existence of a giant component, but its size is not known. The only cases that the existence of a macroscopic component is not yet proven are the ones exactly at the zero-range criticality with $b > 3$. This is because the second moment calculated from the distribution ν_1 exists in this region only, and the critical curve $r_c(\phi)$ was inversely proportional to the second derivative of $Z(\phi)$. Therefore $\lim_{\phi\to 1} r_c(\phi) = 0$ if and only if $b \leqslant 3$.

Next we show that, at the zero-range critical point with $b > 3.51$, our degree sequence is with high probability such that it satisfies the conditions of Molloy and Reed. Evidently, the construction of the lower bound for the probability of the conditioning event follows the lines of the subcritical analysis. The result of Jeon *et al.* in Eq. (37) gives an estimate for $P(A_2 \mid A_1)$ in the whole phase diagram with condensates, that is $b > 2$, but due to the restriction to the region with existing second moment of the measure ν_1, things are settled by the usual local limit theorem. Concerning the upper bounds for the deviations of $n(n-2)C_n/L$ around its mean $n(n-2)\nu_1^r(n)$ (see Appendix A), we see that locally the condition $b > 3.01$ would suffice: The second moment $\sum_{n\geqslant 0} n^2 \nu_1^r(n) < M$ for some constant M and, since the largest degree is $O(L^{1/(b-1)})$ with high probability,

$$P\left(\left|n(n-2)\frac{C_n}{L} - n(n-2)\nu_1^r(n)\right| > \epsilon\right) \leqslant \exp\left(-\frac{\epsilon L}{n^2} f\left(\frac{\epsilon}{M}\right)\right) \longrightarrow 0 \quad (39)$$

for $n = O(L^{1/(b-1)})$ as $L \to \infty$. The magnitude of the error term in calculating

$$\sum_{n\geqslant 1} n(n-2) D_n / L \quad (40)$$

turns out to be crucial. Writing the estimate (13) in a multiplicative form (this is, in fact, exactly the same bound as given in the Appendix A of ref. 26) and using again the inequality $\epsilon f(\epsilon) \geqslant \epsilon^2/(2+\epsilon)$ yields

$$P\left(\left|\frac{C_n}{L} - \nu_1^r(n)\right| > \epsilon \nu_1^r(n)\right) \leqslant \exp\left(-\nu_1^r(n) L \frac{\epsilon^2}{2+\epsilon}\right), \quad (41)$$

so that we get a little sharper estimate than that given by the formula (14):

$$C_n = \nu_1^r(n) L + o\left(\sqrt{\nu_1^r(n)} L^{1/2+\delta}\right). \quad (42)$$

Since $\nu_1^r(n) \leqslant \nu_1(n)/r \sim n^{-b}$ for large n, the error in the quantity (40) is

$$\sum_{n=1}^{O(L^{1/(b-1)})} n^2 \sqrt{\nu_1^r(n)} L^{-1/2+\delta} = O\left(L^{\frac{7-2b}{2(b-1)}+\delta}\right). \tag{43}$$

Thus $b > 3.51$ is enough to get rid of it.

5.1. Phase Diagram and Critical Exponents

The simple approximation by independent degrees fails to give positive results even for the existence of the giant component for all the values of the parameter b. Now, as we start exploring the phase diagram, we present the uncertain cases as conjectures—they will be given some support by numerics exactly at criticality and analysis just below it, suggesting a cross-over between the subcritical and non-trivial behaviors. The curves A–F of Fig. 2 show the critical curves for a set of values of b, each representing a type of diagram different from the others. They will be commented one at a time as we discuss the corresponding values b.

A remark on the calculations: Since many of the results discuss the behavior of the system in the high density limit, in that $\phi \lesssim 1$, the linear

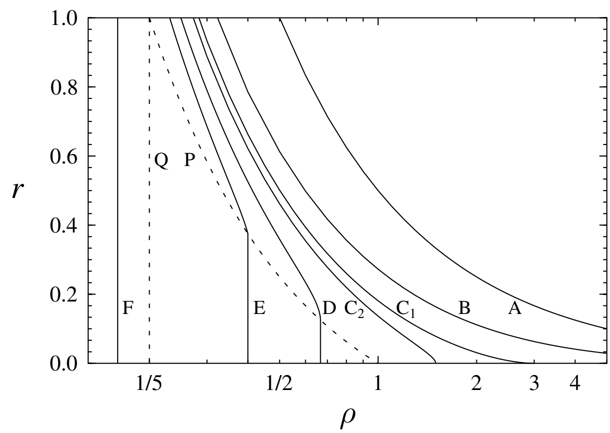

Fig. 2. The curves A–F show phase diagrams for the Evans interaction with various values of b: From A to F, $b=0, 3/2, 7/3, 8/3, 7/2, 9/2$ and 8. The dotted line P is the curve $S(\rho)$ of limiting points for $3<b\leqslant 7$ and the dotted line Q marks the value of density ρ, below which the birth of the giant component is possible only through zero-range condensation.

transformation formula for hypergeometric functions (15.3.6 of ref. 25)

$$\begin{aligned}F(a_1,a_2;c;\phi) = {} & \frac{\Gamma(c)\Gamma(c-a_1-a_2)}{\Gamma(c-a_1)\Gamma(c-a_2)}F(a_1,a_2;a_1+a_2-c+1;1-\phi)\\ & +(1-\phi)^{c-a_1-a_2}\frac{\Gamma(c)\Gamma(a_1+a_2-c)}{\Gamma(a_1)\Gamma(a_2)}\\ & \times F(c-a_1,c-a_2;c-a_1-a_2+1;1-\phi),\end{aligned} \tag{44}$$

valid for $c \neq a_1+a_2+m,\ m\in\mathbb{Z}$, with expansions of the hypergeometric functions on the right-hand side around $\phi=1$, proves to be very useful. It turns out that the transformation formula is not valid for integer b in our model, so that these situations must be dealt with separately by repeatedly using the Gauss' relations for contiguous functions given in ref. 25.

The $1/\rho$-behavior of r_c corresponding to $\mathbf{b=0}$ is plotted as the curve A of Fig. 2. This benchmark case has already been covered in Section 4.1, so we start exploring the phase diagram from non-zero values of b. There are no condensates for $\mathbf{0<b<1}$, so that the results of the subcritical cases apply. Using the above transformation formula and expanding in Eqs. (34) and (36) one gets for high densities that

$$r_c(\rho)\approx\frac{1-b}{(2-b)\rho}, \tag{45}$$

exactly the same as the prediction (32)! According to our calculations in Section 4.2 this should be the only region, where the critical curve decays as a first inverse power of ρ. Indeed, at $\mathbf{b=1}$,

$$Z(\phi)=-(\log(1-\phi))/\phi, \tag{46}$$

leading to a logarithmic correction to the first inverse power law. Furthermore, for $\mathbf{1<b<2}$, condensation does not occur, but we see a drastic change in the behavior of the critical curve,

$$r_c(\rho)\approx\frac{1}{2-b}\left(\frac{(b-1)\Gamma(2-b)\Gamma(b)}{\rho}\right)^{\frac{1}{2-b}}, \tag{47}$$

in that it decays with a b-dependent exponent greater than 1. The curve B in Fig. 2 with $b=3/2$ is an example from this region. The case $\mathbf{b=2}$ involves again logarithmic corrections. We remark that high density approximations for the amplitude Δ of the giant component could have been computed from Eq. (19) in the same manner as for the critical curves. What is important is that, by the subcriticality in the zero-range sense, the growth of the giant component is linear in the vicinity of the phase transition, i.e. the critical exponent for the size of the giant component equals unity.

There is a finite zero-range critical density $\rho_c = 1/(b-2)$ for $\mathbf{2 < b < 3}$, but the second derivative of the partition function $Z(\phi)$, i.e. the second factorial moment of a ν_ϕ-distributed variable, still diverges in the limit $\phi \to 1$. This means that r_c hits zero at $\rho = 1/(b-2)$. Again from the expansions of the hypergeometric functions, we have

$$r_c(\rho) \approx \frac{(b-2)^{\frac{5-2b}{b-2}}}{[(b-1)\Gamma(3-b)\Gamma(b)]^{\frac{1}{b-2}}} \left(\frac{1}{b-2} - \rho\right)^{\frac{3-b}{b-2}}. \tag{48}$$

The curves C_1 (b=7/3) and C_2 (b=8/3) in Fig. 2 with zero-range critical points 3 and 3/2, and high density exponents 2 and 1/2 for the critical curves, respectively, are examples from this region. At the exact zero-range criticality, there is a giant component for any non-zero r, but the lack of proof prevents us from exactly calculating its size. Certainly something interesting is happening: Keeping $\delta = r - r_c(\phi)$ fixed and letting $\phi \to 1$, we have

$$\Delta \approx \frac{2\phi Z''(\phi)^2}{Z(\phi)Z'''(\phi)}\delta \longrightarrow \begin{cases} 0, & \text{for } b \leqslant 4, \\ \frac{8(b-4)}{9(b-2)(b-3)}\delta, & \text{for } b > 4, \end{cases} \tag{49}$$

so that the linear growth is expected to be taken over by some other behavior. To set up a conjecture, we now calculate the leading order approximation as if the conditions of Molloy and Reed were satisfied. So we wish to analyze

$$\beta = 1 - \frac{F(2,2;2+b;(1-\beta r)\phi)}{F(2,2;2+b;\phi)}, \tag{50}$$

$$\Delta = 1 - \frac{F(1,1;1+b;(1-\beta r)\phi)}{F(1,1;1+b;\phi)} \tag{51}$$

at $\phi = 1$. Expansions of the hypergeometric functions to the leading order in β then yield for the size of the giant component the expression

$$\Delta \approx \frac{1}{b-2}\left|\frac{\Gamma(b-2)}{\Gamma(2-b)\Gamma(b)^2}\right|^{\frac{1}{b-3}} r^{\frac{1}{3-b}}, \tag{52}$$

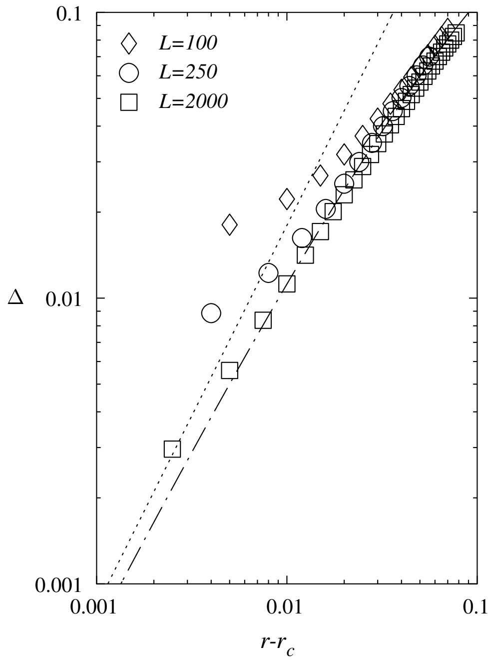

Fig. 3. Simulation results for the size of the giant component at the zero-range critical point for $b = 2.25$. The dash-dotted line is a numerical solution for the Eqs. (16) and (17). The dotted line shows the conjecture (52) for the leading order.

as $r \to 0$, in that the critical exponent would be $1/(3 - b)$. Results of Monte-Carlo simulations, presented in Fig. 3 for $b=2.25$, indicate that the Eqs. (50) and (51) really describe the size of the giant component correctly, although the convergence seems to be quite slow.[27]

To give more support to our conjecture, we remark that it can be shown that the non-trivial term is already present in the expansion of Δ for ϕ just below the radius of convergence. This is obtained by expanding the hypergeometric function in Eq. (50) in such a way that $1-\phi \ll \beta r\phi \ll 1$, in practice meaning that we are sufficiently far away from the critical curve measured in terms of $1-\phi$, and the result is the same as Eq. (52) but with r multiplied by ϕ. Since the coefficient of the linear term vanishes in the limit, we then know that, at least for $b<2.5$, a crossover must exist at some small value of r that vanishes in the limit $\phi \to 1$. The cases $b \geqslant 2.5$ would require a higher order analysis of the size of the giant component below the condensation point in order to make sure that the coefficients of the analytical terms with powers between 1 and $1/(b-3)$ tend to zero as well.

The critical curve for $\mathbf{b}=\mathbf{3}$ is of the same type as for the $2<b<3$ in the sense that it approaches zero as ρ tends to ρ_c. The linear transformation formula being not valid for integer b, one has to deal with the exact expressions for the derivatives of the partition function. This implies the conjecture

$$\Delta \approx \exp\left(-\frac{1}{4r}\right) \tag{53}$$

for the critical growth as $r \to 0$, i.e. an infinite order transition.

Now we enter the region $\mathbf{3}<\mathbf{b}<\mathbf{4}$, nearly half of which allows a rigorous analysis. Strictly speaking, we don't even know the exact position of the phase transition for $b \leqslant 3.5$ at $\phi=1$. This is because the limit of the critical curve r_c as $\phi \to 1$ is greater than zero by the existence of the second moment of a ν_1-distributed variable. More precisely, we have $\lim_{\phi\to 1} r_c(\phi)=(b-3)/4$, this holding for all $b>3$, so that, by $\rho_c=1/(b-2)$, the end points define a curve

$$S(\rho)=\max\left\{0, \frac{1}{4}\left(\frac{1}{\rho}-1\right)\right\} \tag{54}$$

on the (ρ, r)-plane. Below $S(\rho)$, the transition curves are vertical and the giant component can be born only through condensation of edges. The way the curves $r_c(\rho)$ approach the zero-range critical point can again be evaluated,

$$r_c(\rho) \approx \frac{b-3}{4} + \frac{\Gamma(b)^2\Gamma(4-b)}{16(b-3)\Gamma(b-3)}\left[\frac{(b-2)^2(b-3)}{3b-5}\left(\frac{1}{b-2}-\rho\right)\right]^{b-3}. \tag{55}$$

The curve D of Fig. 2 shows an example with $b=7/2$. In the figure, the curve P drawn with a dotted line is $S(\rho)$, the curve of limiting points of r_c. For $\mathbf{3.51}<\mathbf{b}<\mathbf{4}$ we can now prove that, at $\rho=\rho_c=1/(b-2)$ as $r \to r_c(\rho)=(b-3)/4$ from above,

$$\Delta \approx \frac{1}{b-2}\left(\frac{16\Gamma(b-2)}{(b-3)^2\Gamma(b)^2\Gamma(2-b)}\right)^{\frac{1}{b-3}}\left(r-\frac{b-3}{4}\right)^{\frac{1}{b-3}}, \tag{56}$$

thus giving the exponent $1/(b-3)$, exactly the inverse of the exponent for the critical curve. Remember that for $2<b<3$ we conjectured $1/(3-b)$ for

Δ but $(3-b)/(b-2)$ for $r_c(\rho)$, so that no such relation is valid for small values of b. The case $\mathbf{b}=\mathbf{4}$ again involves logarithms.

We have two types of critical behavior to deal with left. The first one, occurring in the region $\mathbf{4}<\mathbf{b}<\mathbf{7}$, can be seen as a continuation to the preceding interval, since the endpoints of r_c lie on the curve $S(\rho)$ with values strictly less than one because $\rho_c > 1/5$. The change is in the exponents: Near the zero-range criticality we have

$$r_c(\rho) \approx \frac{b-3}{4} + \left(\frac{9(b-3)}{4(b-4)} - 1\right) \frac{(b-2)^2(b-3)}{3b-5} \left(\frac{1}{b-2} - \rho\right), \tag{57}$$

in that r_c forms a cusp with the vertical line at ρ_c (see the curve E with $b=9/2$ in Fig. 2). Moreover, we recover the linear growth of the giant component, that we predicted already from the subcritical analysis in formula (49),

$$\Delta \approx \frac{8(b-4)}{9(b-2)(b-3)} \left(r - \frac{b-3}{4}\right). \tag{58}$$

The region in the phase diagram that we have not yet discussed is $\mathbf{b}\geqslant\mathbf{7}$. But this is trivial, since $S(\rho)\geqslant 1$ for all values $\rho \leqslant \rho_c$, and the phase transition curves are vertical lines positioned at the critical density of the zero-range process (curve F in Fig. 2). In other words, no giant component can exist without a condensate of edge ends on one of the vertices.

We remark that the critical exponents for the growth of the giant component at the condensation point are exactly the same as obtained recently by Cohen *et al.*[(28)] for diluted scale-free networks. This is not a coincidence: In their model, from a random graph with a given power-law degree distribution, a fraction of the vertices with all the edges joined to them are removed. For the vertices retained in the graph this means that their degree distribution is transformed exactly in the same manner as the distribution for the vertices in W in the operation of removing the dangling ends from our bipartite graph. The distribution ν_1 corresponding to the critical density being heavy-tailed, the similarity between the two problems is obvious. In the dynamical model, however, we have to deal with the fluctuations and, as the analysis shows, they are not always easily controlled.

6. CONCLUSIONS

We studied the existence and size of the giant component on one of the vertex sets in a dynamical model for bipartite multigraphs using the results of Molloy and Reed.[12,13] Motivated by the concept of preferential attachment, we defined the movement of the edge ends on one side of the bipartite graph to be a zero-range process with jump rates bounded away from zero, in the stationary state of which only weak correlations exist. A simple approximation by independent random variables allowed us to bound the fluctuations, and show that the degree sequences obtained this way are well-behaved in the sense of Molloy and Reed for zero-range processes with density below the critical value for condensation. As a benchmark, it was shown that the case of non-interacting edge ends produced the well-known results of the $G_{n,M}$-model of random graphs. Generally, we find four types of critical curves, depending on the asymptotics of the derivatives of the grand canonical partition function near the radius of convergence. A class of rate functions, for which the curves are of the same type as for the independent walkers, was also identified. The last sections of the article were devoted to the Evans interaction, showing all the four types of phase diagrams as the parameter of the interaction is varied. Our simple use of the conditioning device was strong enough to access some nontrivial parts of the parameter space but did not allow us to show that the conditions of Molloy and Reed are satisfied for all values of the interaction parameter exactly at the zero-range condensation point. The critical exponents for the growth of the giant component were given as conjectures supported by Monte-Carlo simulations in those cases. In the end, we discussed the connection with the model of diluted scale-free networks.[28] This implies that our model with the Evans interaction and exactly at the zero-range condensation point belongs to the universality class of percolation on graphs with power-law distributed degrees. The same set of exponents appears also in ref. 29, where a general, phenomenological Landau theory for phase transitions on scale-free networks is constructed. The simulation results suggest that the simple Landau theory is not broken by the fluctuations in our model.

Finally, we would like to mention a few points that would be interesting to analyze further. First of all, our simple approximations by independent variables failed to give estimates sharp enough to completely cover the parts of the parameter space with condensates for the Evans interaction. We believe that the problem is purely technical and could be removed by more sophisticated use of the conditioning device. Secondly, no results "inside" the phase transition, i.e. no further than $o(1)$ from the critical point, exist for graphs with a general degree sequence. Therefore, the behavior near $r = r_c$ remains to be discovered. What comes to the underlying particle system, there are still open questions even concerning the statics of zero-range processes, especially for rate functions exhibiting slow decay.[16] Furthermore, only stationary properties of the model were discussed in this article. Recently, advances have been made towards understanding the formation of condensates in zero-range processes as a function of time,[30,15] so that one is tempted to consider the temporal scalings for graph-valued processes as well. A grand canonical generalization of the present model that would allow for a variable number of external agents would also be of interest.

APPENDIX A. DEFINITIONS FROM REFS. 12, 13

We give the definitions of Molloy and Reed in our notation. An asymptotic degree sequence is a sequence of integer-valued functions $\mathcal{D} = D_0(L), D_1(L), \ldots$ such that $D_n(L) = 0$, for $n \geqslant L$ and $\sum_{n\geqslant 0} D_n(L) = L$.

Given an asymptotic degree sequence $\mathcal{D}$, $\mathcal{D}_L$ is set to be the degree sequence $c_1, c_2, \ldots, c_L$, where $c_j \geqslant c_{j+1}$ and $|j : c_j = n| = D_n(L)$. Let $\Omega_{\mathcal{D}_L}$ be the set of graphs with vertex set $1, \ldots, L$ with degree sequence $\mathcal{D}_L$.

An asymptotic degree sequence $\mathcal{D}$ is well-behaved if:

1. $\mathcal{D}$ is feasible and smooth, i.e. $\Omega_{\mathcal{D}_L} \neq \emptyset$ and there are constants λ_n such that $\lim_{L\to\infty} D_n(L)/L = \lambda_n$.

2. $n(n-2)D_n(L)/L$ tends uniformly to $n(n-2)\lambda_n$, i.e. for all $\epsilon > 0$ there exists L_ϵ such that for all $L > L_\epsilon$ and for all $n \geqslant 0$

$$\left| \frac{n(n-2)D_n(L)}{L} - n(n-2)\lambda_n \right| < \epsilon$$

3. $$\mathcal{L}(\mathcal{D}) = \lim_{L\to\infty} \sum_{n\geqslant 1} n(n-2)D_n(L)/L$$

exists, and the sum approaches the limit uniformly, i.e.:

(a) If $\mathcal{L}(\mathcal{D})$ is finite then for all $\epsilon > 0$, there exist n^*, L_ϵ such that for all $L > L_\epsilon$:

$$\left| \sum_{n=1}^{n^*} \frac{n(n-2)D_n(L)}{L} - \mathcal{L}(\mathcal{D}) \right| < \epsilon$$

(b) If $\mathcal{L}(\mathcal{D})$ is infinite then for all $T > 0$, there exist n^*, L_ϵ such that for all $L > L_\epsilon$:

$$\left| \sum_{n=1}^{n^*} \frac{n(n-2)D_n(L)}{L} \right| > T.$$

Furthermore, $\mathcal{D}$ is called sparse if $\sum nD_n(L)/L = K + o(1)$ for some constant K.

In the theorems of Molloy and Reed, $\mathcal{D}$ is assumed to be a well-behaved sparse asymptotic degree sequence with the property that there is an $\epsilon > 0$ such that for all L and $n > L^{1/4-\epsilon}$, $D_n = 0$. In proving some statements concerning the cycles $1/4 - \epsilon$ has to be replaced by more restrictive $1/8 - \epsilon$. The figure 1/4 is needed for the multigraph of a random configuration to be simple, and is therefore not required in our analysis. In fact only $1/2 - \epsilon$, which is high enough for our purposes, is needed for the statements about the existence and size of the giant component to hold.

Notice that our sequence $\mathcal{D}$ defined in Section 3 cannot satisfy the conditions almost surely (i.e. with probability equal to 1) but with high probability only. This does not matter, since all the results are given in the probabilistic setting anyway, and basically means a one more choice of L_ϵ in the proofs.

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27. Our main theoretical results and conjectures were tested by performing Monte-Carlo simulations, where the random number generator Mersenne twister was utilized. For the less interesting small values of ρ the simulations were easier than for high density, where to actually observe the leading order critical behavior (and the exponent) would require simulations with of the order of hundred-thousand vertices and even a larger number of particles on W. Note that the parameter M_2 should be a "macroscopic" number, which becomes a problem for $r=2M_2/N$ approaching zero as in the case of Fig. 3.
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